

Fidelity of Single Qubit Maps

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We give a simple way of characterising the average fidelity between a unitary and a general operation on a single qubit which only involves calculating the fidelities for a few pure input states.

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Consider transforming a quantum system by a desired operator, U . In practice, we may not be able to implement U exactly but may actually apply a map \mathcal{S} instead. In general, the map \mathcal{S} is a superoperator. To gauge how closely \mathcal{S} approximates U (or *vice versa*), we require a measure of how close the output states are to each other, *i.e.* given identical input states, ρ_{in} , how does $U[\rho_{in}]$ compare with $\mathcal{S}[\rho_{in}]$. For the most general case where both output states are given by density operators, we can define their fidelity as [1],

$$F(\rho_1, \rho_2) = \left(\text{Tr} \left(\sqrt{\sqrt{\rho_1} \rho_2 \sqrt{\rho_1}} \right) \right)^2, \quad (1)$$

which simplifies to

$$F(|\psi\rangle\langle\psi|, \rho) = \text{Tr}(|\psi\rangle\langle\psi| \rho), \quad (2)$$

in the case where at least one of the states is pure [2]. This situation arises naturally when U is unitary and the input state, ρ_{in} , is pure, as in this case $U[\rho_{in}]$ will also be pure.

In the case where our system is a single qubit, the allowed operations are restricted to affine contractions of the Bloch Ball. We will consider the case where U is

a desired unitary transformation (rotation) of the Bloch ball, and \mathcal{S} is a superoperator. It is assumed that \mathcal{S} is a linear, trace-preserving map on the space of single qubit density operators [3]. We will define \bar{F} as the average fidelity over all pure input states,

$$\bar{F} = \frac{1}{4\pi} \int F_{|\psi\rangle\langle\psi|} d\Omega, \quad (3)$$

where

$$F_{|\psi\rangle\langle\psi|} = \text{Tr}(U|\psi\rangle\langle\psi|U^\dagger \mathcal{S}[|\psi\rangle\langle\psi|]). \quad (4)$$

The state $|\psi\rangle\langle\psi|$ can be expressed in the basis of the Pauli spin matrices

$$\begin{aligned} |\psi\rangle\langle\psi| &= \frac{1}{2} \left(\mathbf{1} + \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \right) \\ &= \frac{\sigma_0}{2} + \sin\theta \cos\phi \frac{\sigma_x}{2} + \sin\theta \sin\phi \frac{\sigma_y}{2} + \cos\theta \frac{\sigma_z}{2} \\ &= \sum_{j=0,x,y,z} c_j(\theta, \phi) \frac{\sigma_j}{2}. \end{aligned} \quad (5)$$

Hence Eq. (3) can be expressed as

$$\begin{aligned} \bar{F} &= \frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \text{Tr} \left(U \sum_j c_j(\theta, \phi) \frac{\sigma_j}{2} U^\dagger \mathcal{S} \left[\sum_k c_k(\theta, \phi) \frac{\sigma_k}{2} \right] \right) \sin\theta d\phi d\theta \\ &= \sum_{jk} \left(\frac{1}{4\pi} \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} c_j c_k \sin\theta d\theta d\phi \right) \text{Tr} \left(U \frac{\sigma_j}{2} U^\dagger \mathcal{S} \left[\frac{\sigma_k}{2} \right] \right) \end{aligned} \quad (6)$$

where we have used the linearity of U and \mathcal{S} . The integrals of the coefficients, c_{jk} , are easily evaluated by symmetry: the cross terms vanish [4] leaving the simpler

expression

$$\begin{aligned} \bar{F} &= \sum_{jk} \left(\frac{2\delta_{j0}\delta_{k0} + \delta_{jk}}{3} \right) \text{Tr} \left(U \frac{\sigma_j}{2} U^\dagger \mathcal{S} \left[\frac{\sigma_k}{2} \right] \right) \\ &= \text{Tr} \left(U \frac{\sigma_0}{2} U^\dagger \mathcal{S} \left[\frac{\sigma_0}{2} \right] \right) + \frac{1}{3} \sum_{j=x,y,z} \text{Tr} \left(U \frac{\sigma_j}{2} U^\dagger \mathcal{S} \left[\frac{\sigma_j}{2} \right] \right) \\ &= \frac{1}{2} + \frac{1}{3} \sum_{j=x,y,z} \text{Tr} \left(U \frac{\sigma_j}{2} U^\dagger \mathcal{S} \left[\frac{\sigma_j}{2} \right] \right) \end{aligned} \quad (7)$$

where we have used the fact that $\mathcal{S}[\sigma_0/2]$ is a density matrix and thus has unit trace. In order to express the average fidelity in terms of states, and to give a more intuitive picture of the above expression, we use the substitutions,

$$\begin{aligned}\frac{\sigma_j}{2} &= \frac{\sigma_0 + \sigma_j}{2} - \frac{\sigma_0}{2} = \rho_j - \rho_0 \\ &= \frac{\sigma_0}{2} - \frac{\sigma_0 - \sigma_j}{2} = \rho_0 - \rho_{-j},\end{aligned}\quad (8)$$

where $\rho_{\pm j}$ represents a pure state in the $\pm j$ -direction and ρ_0 is the maximally mixed state. This gives the two

equivalent expressions,

$$\bar{F} = \frac{1}{2} + \frac{1}{3} \sum_{j=x,y,z} \text{Tr} (U \rho_j U^\dagger \mathcal{S}[\rho_j] - U \rho_j U^\dagger \mathcal{S}[\rho_0]) \quad (9)$$

$$\bar{F} = \frac{1}{2} + \frac{1}{3} \sum_{j=x,y,z} \text{Tr} (U \rho_{-j} U^\dagger \mathcal{S}[\rho_{-j}] - U \rho_{-j} U^\dagger \mathcal{S}[\rho_0]) \quad (10)$$

and taking their average yields,

$$\begin{aligned}\bar{F} &= \frac{1}{2} + \frac{1}{6} \sum_{j=x,y,z} \text{Tr} (U \rho_j U^\dagger \mathcal{S}[\rho_j] + U \rho_{-j} U^\dagger \mathcal{S}[\rho_{-j}] - U(\rho_j + \rho_{-j}) U^\dagger \mathcal{S}[\rho_0]) \\ &= \frac{1}{2} + \frac{1}{6} \sum_{j=x,y,z} \text{Tr} (U \rho_j U^\dagger \mathcal{S}[\rho_j] + U \rho_{-j} U^\dagger \mathcal{S}[\rho_{-j}] - 2 U \rho_0 U^\dagger \mathcal{S}[\rho_0]) \\ &= \frac{1}{2} + \frac{1}{6} \sum_{j=x,y,z} (\text{Tr} (U \rho_j U^\dagger \mathcal{S}[\rho_j]) + \text{Tr} (U \rho_{-j} U^\dagger \mathcal{S}[\rho_{-j}]) - 1) \\ &= \frac{1}{6} \sum_{j=\pm x, \pm y, \pm z} (\text{Tr} (U \rho_j U^\dagger \mathcal{S}[\rho_j])).\end{aligned}\quad (11)$$

Hence, the fidelity of the superoperator \mathcal{S} with the unitary operator U can be calculated by simply averaging the fidelities of the six axial pure states on the Bloch sphere, $\{\rho_{+x}, \rho_{-x}, \rho_{+y}, \rho_{-y}, \rho_{+z}, \rho_{-z}\}$. In the case where \mathcal{S} is unital ($\mathcal{S}[\rho_0] = \rho_0$) it can be seen from Eq. 9 that this reduces to an average over only three states, $\{\rho_{+x}, \rho_{+y}, \rho_{+z}\}$; similarly Eq. 10 shows that the average can be taken over $\{\rho_{-x}, \rho_{-y}, \rho_{-z}\}$.

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